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Analytic Solutions of a Second-Order Iterative Functional Differential Equation

JIAN-GUO SI AND XIN-PING WANG
Department of Mathematics, Binzhou Normal College
Binzhou, Shandong 256604, P.R. China

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Abstract—This paper is concerned with a second-order iterative functional differential equation $x''(z) = (x^m(z))^2$. Its analytic solutions are discussed by locally reducing the equation to another functional differential equation with proportional delays $\mu^2 y''(\mu z)y'(z) = \mu y'(\mu z)y''(z) + [y'(z)]^3[y(\mu^m z)]^2$ and by constructing a convergent power series solutions for the latter equation.

Keywords—Functional differential equation, Analytic solution.

1. INTRODUCTION

In the last few years, there has been a growing interest in studying the existence of solutions of functional differential equations with state dependent delay. We refer the reader to the papers by Cooke [1], Eder [2], Feckan [3], Stanek [4], Grimm [5], Driver [6], Oberg [7], Jackiewicz [8], Dunkel [9], Wang [10], and the authors of [11–17]. However, there are only few papers dealing with a second-order iterative functional differential equation. In [18], Petahov considered the existence of solutions of the equation

$$x''(t) = ax(x(t)).$$

In [12], the authors studied the existence of analytic solutions of the equation

$$x''(z) = x\left(az + bx'(z)\right).$$

The purpose of this paper is to discuss the existence of analytic solutions of a second-order iterative functional differential equation of the form

$$x''(z) = \left(x^m(z)\right)^2, \tag{1}$$

where $x^m(z)$ denotes the m^{th} iterate of the function $x(z)$.

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The general idea for our discussion is similar to that used in [12]. To find analytic solutions of (1), we locally reduce equation (1) to another functional differential equation with proportional delays (the research of such equations with proportional delay is in itself very interesting, see [19–22])

$$\mu^2 y''(\mu z)y'(z) = \mu y'(\mu z)y''(z) + [y'(z)]^3 [y(\mu^m z)]^2 \tag{2}$$

satisfying the condition

$$y(0) = s, \qquad y'(0) = \eta \neq 0, \tag{3}$$

called the auxiliary equation of (1), where μ satisfies one of the following conditions:

(H1) $0 < |\mu| < 1$;

(H2) $|\mu| = 1, \mu$ is not a root of unity, and

$$\log \frac{1}{|\mu^n - 1|} \leq K \log n, \qquad n = 2, 3, \dots,$$

for some positive constant K. Then we show that

$$x(z) = y\left(\mu y^{-1}(z)\right)$$

is an analytic solution of (1) in a neighborhood of s. Here $y^{-1}(z)$ denotes the inverse function of $y(z)$.

2. SOME PREPARATORY LEMMAS

In this section, we will state and prove some preparatory lemmas which will be used in the proof of our main result.

LEMMA 1. *Assume that (H1) holds. Then for the initial conditions (3), equation (2) has an analytic solution of the form*

$$y(z) = s + \eta z + \sum_{n=2}^{\infty} b_n z^n \tag{4}$$

in a neighborhood of the origin.

PROOF. Rewrite (2) in the form

$$\frac{\mu y''(\mu z)y'(z) - y'(\mu z)y''(z)}{[y'(z)]^2} = \frac{1}{\mu} y'(z) [y(\mu^m z)]^2$$

or

$$\left(\frac{y'(\mu z)}{y'(z)}\right)' = \frac{1}{\mu} y'(z) [y(\mu^m z)]^2 .$$

Therefore, in view of $y'(0) = \eta \neq 0$, we have

$$y'(\mu z) = y'(z) \left[1 + \frac{1}{\mu} \int_0^z y'(t) \left(y(\mu^m t)\right)^2 dt\right]. \tag{5}$$

We now seek a solution of (2) in the form a power series (4). By defining $b_0 = s$ and $b_1 = \eta$ and substituting (4) into (5), we see that the sequence $\{b_n\}_{n=2}^{\infty}$ is successively determined by the condition

$$\left(\mu^{n+1} - 1\right)(n+2)b_{n+2} = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} \frac{(i+1)(j+1)\mu^{m(n-i-j)}}{n-i+1} b_{i+1}b_{j+1}b_k b_{n-i-j-k}, \qquad n = 0, 1, 2, \dots, \tag{6}$$

in a unique manner. We need to show that the power series (4) converges in a neighborhood of the origin. First of all, since

$$\left| \frac{(i+1)(j+1)\mu^{m(n-i-j)}}{(\mu^{n+1}-1)\,(n+2)(n-i+1)} \right| \le \frac{1}{|\mu^{n+1}-1|} \le M,$$

for some positive number M, thus, if we define a sequence $\{D_n\}_{n=0}^{\infty}$ by $D_0 = |s|$, $D_1 = |\eta|$, and

$$D_{n+2} = M \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} D_{i+1} D_{j+1} D_k D_{n-i-j-k}, \qquad n = 0, 1, \dots,$$

then in view of (6),

$$|b_n| \le D_n, \qquad n = 0, 1, \dots.$$

Now, if we define

$$G(z) = \sum_{n=0}^{\infty} D_n z^n, \tag{7}$$

then

$$G^2(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} D_k D_{n-k} \right) z^n,$$

$$\begin{aligned} G^3(z) &= \left(|s| + \sum_{n=0}^{\infty} D_{n+1} z^{n+1} \right) \left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} D_k D_{n-k} \right) z^n \right) \\ &= |s| \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} D_k D_{n-k} \right) z^n + \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \sum_{k=0}^{n-j} D_{j+1} D_k D_{n-j-k} \right) z^{n+1}, \end{aligned}$$

and

$$\begin{aligned} G^4(z) &= \left(|s| + \sum_{n=0}^{\infty} D_{n+1} z^{n+1} \right) \left(|s| \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} D_k D_{n-k} \right) z^n \right. \\ &\qquad \left. + \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \sum_{k=0}^{n-j} D_{j+1} D_k D_{n-j-k} \right) z^{n+1} \right) \\ &= |s|^2 \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} D_k D_{n-k} \right) z^n \\ &\quad + 2|s| \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \sum_{k=0}^{n-j} D_{j+1} D_k D_{n-j-k} \right) z^{n+1} \\ &\quad + \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} D_{i+1} D_{j+1} D_k D_{n-i-j-k} \right) z^{n+2} \\ &= |s|^2 G^2(z) + 2|s| \left(G^3(z) - |s| G^2(z) \right) + \frac{1}{M} \sum_{n=0}^{\infty} D_{n+2} z^{n+2} \\ &= |s|^2 G^2(z) + 2|s| \left(G^3(z) - |s| G^2(z) \right) + \frac{1}{M} \left(G(z) - |s| - |\eta| z \right) \\ &= 2|s| G^3(z) - |s|^2 G^2(z) + \frac{1}{M} G(z) - \frac{1}{M} \left(|\eta| z + |s| \right), \end{aligned}$$

that is,

$$G^4(z) - 2|s|G^3(z) + |s|^2G^2(z) - \frac{1}{M}G(z) + \frac{1}{M}(|\eta|z + |s|) = 0. \tag{8}$$

Let

$$R(z,\omega) = \omega^4 - 2|s|\omega^3 + |s|^2\omega^2 - \frac{1}{M}\omega + \frac{1}{M}(|\eta|z + |s|),$$

for (z,ω) from a neighborhood of $(0,|s|)$. Since $R(0,|s|) = 0, R'_\omega(0,|s|) = -(1/M) \neq 0$, there exists a unique function $\omega(z)$, analytic on a neighborhood of zero, such that $\omega(0) = |s|, \omega'(0) = |\eta|$ and satisfying the equality $R(z,\omega(z)) = 0$. According to (7) and (8), we have $G(z) = \omega(z)$. It follows that the power series (7) converges on a neighborhood of the origin, which implies that the power series (4) also converges in a neighborhood of the origin. The proof is complete. ∎

Now we introduce the following lemma, the proof of which can be found in [23, Chapter 6] or [24, pp. 166–174].

LEMMA 2. *Assume that (H2) holds. Then there is a positive number δ such that $|\mu^n - 1|^{-1} < (2n)^\delta$ for $n = 1,2,\ldots$. Furthermore, the sequence $\{d_n\}_{n=1}^\infty$ defined by $d_1 = 1$ and*

$$d_n = \frac{1}{|\mu^{n-1} - 1|} \max_{\substack{n=n_1+\cdots+n_t,\\ 0<n_1\le\cdots\le n_t, t\ge 2}} \{d_{n_1}\cdots d_{n_t}\}, \qquad n = 2,3,\ldots$$

will satisfy

$$d_n \le \left(2^{5\delta+1}\right)^{n-1} n^{-2\delta}, \qquad n = 1,2,\ldots.$$

LEMMA 3. *Suppose (H2) holds. Then for $0 < |\eta| \le 1$, equation (2) has an analytic solution of the form*

$$y(z) = s + \eta z + \sum_{n=2}^\infty b_n z^n \tag{9}$$

in a neighborhood of the origin.

PROOF. As in the proof of Lemma 1, we seek a power series solution of form (9). Then defining $b_0 = s$ and $b_1 = \eta$, (6) holds again. So that

$$|b_{n+2}| \le |\mu^{n+1} - 1|^{-1} \sum_{i=0}^{n}\sum_{j=0}^{n-i}\sum_{k=0}^{n-i-j} |b_{i+1}||b_{j+1}||b_k||b_{n-i-j-k}|, \qquad n = 0,1,\ldots. \tag{10}$$

Let us now consider the equation

$$W(z,\omega) = \omega^4 - 2|s|\omega^3 + |s|^2\omega^2 - \omega + |s| + z = 0, \tag{11}$$

for (z,ω) from a neighborhood of $(0,|s|)$. Since $W(0,|s|) = 0$ and $W'_\omega(0,|s|) = -(1/M) \neq 0$, there exists a unique function $\omega(z)$, analytic on a neighborhood of zero, such that $\omega(0) = |s|, \omega'(0) = 1$ and satisfying the equality $W(z,\omega(z)) = 0$. Now if

$$\omega(z) = |s| + z + \sum_{n=2}^\infty B_n z^n, \tag{12}$$

where the coefficient sequence $\{B\}_{n=0}^\infty$ satisfies $B_0 = |s|, B_1 = 1$ and

$$B_{n+2} = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\sum_{k=0}^{n-i-j} B_{i+1}B_{j+1}B_kB_{n-i-j-k}, \qquad n = 0,1,\ldots, \tag{13}$$

then

$$\omega^2(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} B_k B_{n-k} \right) z^n,$$

$$\begin{aligned}\omega^3(z) &= \left(|s| + \sum_{n=0}^{\infty} B_{n+1} z^{n+1} \right) \left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} B_k B_{n-k} \right) z^n \right) \\ &= |s| \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} B_k B_{n-k} \right) z^n + \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \sum_{k=0}^{n-j} B_{j+1} B_k B_{n-j-k} \right) z^{n+1},\end{aligned}$$

and

$$\begin{aligned}\omega^4(z) &= \left(|s| + \sum_{n=0}^{\infty} B_{n+1} z^{n+1} \right) \left(|s| \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} B_k B_{n-k} \right) z^n \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \sum_{k=0}^{n-j} B_{j+1} B_k B_{n-j-k} \right) z^{n+1} \right) \\ &= |s|^2 \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} B_k B_{n-k} \right) z^n \\ &\quad + 2|s| \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \sum_{k=0}^{n-j} B_{j+1} B_k B_{n-j-k} \right) z^{n+1} \\ &\quad + \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} B_{i+1} B_{j+1} B_k B_{n-i-j-k} \right) z^{n+2} \\ &= |s|^2 \omega^2(z) + 2|s| \left(\omega^3(z) - |s| \omega^2(z) \right) + \sum_{n=0}^{\infty} B_{n+2} z^{n+2} \\ &= |s|^2 \omega^2(z) + 2|s| \left(\omega^3(z) - |s| \omega^2(z) \right) + \omega(z) - |s| - z \\ &= 2|s| \omega^3(z) - |s|^2 \omega^2(z) + \omega(z) - |s| - z,\end{aligned}$$

that is, $\omega(z)$ satisfies equation (11). It follows that the power series (12) converges in a neighborhood of zero, and there is a positive constant T such that

$$B_n \leq T^n, \qquad n = 1, 2, \ldots. \tag{14}$$

Now by induction, we prove that

$$|b_n| \leq B_n d_n, \qquad n = 1, 2, \ldots,$$

where the sequence $\{d_n\}_{n=1}^{\infty}$ is defined in Lemma 2. In fact,

$$\begin{aligned}|b_1| &= |\eta| \leq 1 = C_1 d_1, \\ |b_2| &= |\mu - 1|^{-1} |b_1| |b_1| |b_0| |b_0| \\ &\leq |\mu - 1|^{-1} B_1 d_1 \cdot B_1 d_1 \cdot B_0 \cdot B_0 \\ &= B_2 |\mu - 1|^{-1} \max_{\substack{n_1 + n_2 = 2; \\ 0 < n_1 \leq n_2}} \{ d_{n_1} d_{n_2} \} \\ &= B_2 d_2.\end{aligned}$$

Assume that the above inequality holds for $n = 1, 2, \dots, l$. Then

$$\begin{aligned}
|b_{l+1}| &\le |\mu^l - 1|^{-1} \sum_{i=0}^{l-1} \sum_{j=0}^{l-1-i} \sum_{k=0}^{l-1-i-j} |b_{i+1}||b_{j+1}||b_k||b_{l-1-i-j-k}| \\
&= |\mu^l - 1|^{-1} \left(2 \sum_{i=0}^{l-1} |b_{i+1}||b_{j+1}||b_0||b_0| + 2 \sum_{i=0}^{l-1} \sum_{j=0}^{l-2-i} |b_{i+1}||b_{j+1}||b_0||b_{l-1-i-j}| \right. \\
&\quad \left. + \sum_{i=0}^{l-1} \sum_{j=0}^{l-1-i} \sum_{k=1}^{l-2-i-j} |b_{i+1}||b_{j+1}||b_k||b_{l-1-i-j-k}| \right) \\
&\le |\mu^l - 1|^{-1} \left(2 \sum_{i=0}^{l-1} B_{i+1} d_{i+1} B_{j+1} d_{j+1} B_0 B_0 \right. \\
&\quad + 2 \sum_{i=0}^{l-1} \sum_{j=0}^{l-2-i} B_{i+1} d_{i+1} B_{j+1} d_{j+1} B_0 B_{l-1-i-j} d_{l-1-i-j} \\
&\quad \left. + \sum_{i=0}^{l-1} \sum_{j=0}^{l-1-i} \sum_{k=1}^{l-2-i-j} B_{i+1} d_{i+1} B_{j+1} d_{j+1} B_k d_k B_{l-1-i-j-k} d_{l-1-i-j-k} \right) \\
&\le |\mu^l - 1|^{-1} \max_{\substack{n_1+n_2+\cdots+n_t=l+1; \\ 0<n_1\le\cdots\le n_t, t\ge 2}} \{d_{n_1} \dots d_{n_t}\} \left(2 \sum_{i=0}^{l-1} B_{i+1} B_{j+1} B_0 B_0 \right. \\
&\quad \left. + 2 \sum_{i=0}^{l-1} \sum_{j=0}^{l-2-i} B_{i+1} B_{j+1} B_0 B_{l-1-i-j} + \sum_{i=0}^{l-1} \sum_{j=0}^{l-1-i} \sum_{k=1}^{l-2-i-j} B_{i+1} B_{j+1} B_k B_{l-1-i-j-k} \right) \\
&= B_{l+1} d_{l+1}
\end{aligned}$$

as desired. In view of (14) and Lemma 2, we finally see that

$$|b_n| \le T^n \left(2^{5\delta+1}\right)^{n-1} n^{-2\delta}, \qquad n = 1, 2, \dots,$$

that is,

$$\lim_{n\to\infty} \sup (|b_n|)^{1/n} \le \lim_{n\to\infty} \sup T \left(2^{5\delta+1}\right)^{(n-1)/n} n^{-2\delta/n} = T \left(2^{5\delta+1}\right).$$

This implies that the series (9) converges for $|z| < (T(2^{5\delta+1}))^{-1}$. The proof is complete. ∎

3. EXISTENCE OF ANALYTIC SOLUTIONS OF EQUATION (1)

In this section, we state and prove our main result in this note.

THEOREM 1. *Suppose the conditions of Lemma 1 or Lemma 3 are satisfied. Then equation (1) has an analytic solution $x(z)$ in a neighborhood of s. This solution has the form $x(z) = y(\mu y^{-1}(z))$, where $y(z)$ is an analytic solution of the initial problem (2),(3).*

PROOF. By Lemma 1 and Lemma 3, we can find an analytic solution $y(z)$ of the auxiliary equation (2) in the from of (4) such that $y(0) = s$ and $y'(0) = \eta \ne 0$. Clearly, the inverse $y^{-1}(z)$ exists and is analytic in a neighborhood of $y(0) = s$. Let

$$x(z) = y\left(\mu y^{-1}(z)\right), \tag{15}$$

which is also analytic in a neighborhood of s. From (2), it is easy to see

$$\begin{aligned}
x'(z) &= \mu y'\left(\mu y^{-1}(z)\right)\left(y^{-1}(z)\right)' = \frac{\mu y'\left(\mu y^{-1}(z)\right)}{y'\left(y^{-1}(z)\right)}, \\
x''(z) &= \frac{\mu^2 y''\left(\mu y^{-1}(z)\right) - \mu y'\left(\mu y^{-1}(z)\right) y''\left(y^{-1}(z)\right) \cdot \left(1/y'\left(y^{-1}(z)\right)\right)}{\left(y'\left(y^{-1}(z)\right)\right)^2} \\
&= \frac{\mu\left[\mu y''\left(\mu y^{-1}(z)\right) y'\left(y^{-1}(z)\right) - y'\left(\mu y^{-1}(z)\right) y''\left(y^{-1}(z)\right)\right]}{\left[y'\left(y^{-1}(z)\right)\right]^3} \\
&= \left[y\left(\mu^m y^{-1}(z)\right)\right]^2 = \left(x^m(z)\right)^2
\end{aligned}$$

as required. The proof is complete. ∎

In the above theorem, we have shown that under the conditions of Lemma 1 or Lemma 3, equation (1) has an analytic solution $x(z) = y(\mu y^{-1}(z))$ in a neighborhood of the number s, where y is an analytic solution of (2). Since the function $y(z)$ in (11) can be determined by (6), it is possible to calculate, at least in theory, the explicit form of $x(z)$. However, knowing that an analytic solution of (1) exists, we can take an alternative route as follows. Assume that $x(z)$ is of the form

$$x(z) = x(s) + x'(s)(z-s) + \frac{x''(s)}{2!}(z-s)^2 + \cdots,$$

we need to determine the derivatives $x^{(n)}(s)$, $n = 0, 1, \ldots$. First of all, in view of (15), we have

$$x(s) = y\left(\mu y^{-1}(s)\right) = y(\mu \cdot 0) = y(0) = s$$

and

$$x'(s) = \frac{\mu y'\left(\mu y^{-1}(s)\right)}{y'\left(y^{-1}(s)\right)} = \frac{\mu y'(\mu \cdot 0)}{y'(0)} = \mu,$$

respectively. Furthermore,

$$\begin{aligned}
x''(s) &= \left(x^m(s)\right)^2 = s^2. \\
x'''(s) &= 2x^m(s)x'\left(x^{m-1}(s)\right) \ldots x'(x(s))x'(s) \\
&= 2s\mu^m.
\end{aligned}$$

For convenience, we take the following notations:

$$x_{ij}(z) = x^{(i)}\left(x^j(z)\right).$$

By induction, we may prove that

$$\left(x^m(z)\right)^{(k)} = P_{m,k}\left(x_{10}(z), \ldots, x_{1,m-1}(z); \ldots; x_{k0}(z), \ldots, x_{k,m-1}(z)\right),$$

where $P_{m,k}$ is a uniquely defined multivariate polynomial with nonnegative coefficients. Next by calculating the higher derivatives of both sides of (1), we can obtain

$$\begin{aligned}
x^{(3+n)}(z) &= 2\left(x^m(z)\left(x^m(z)\right)'\right)^{(n)} \\
&= 2\sum_{k=0}^{n} C_n^k \left(x^m(z)\right)^{(k)} \left(x^m(z)\right)^{(n-k+1)} \\
&= 2\sum_{k=0}^{n} \left[C_n^k P_{m,k}\left(x_{10}(z), \ldots, x_{1,m-1}(z); \ldots; x_{k0}(z), \ldots, x_{k,m-1}(z)\right)\right. \\
&\quad \left. \times P_{m,n-k+1}\left(x_{10}(z), \ldots, x_{1,m-1}(z); \ldots; x_{n-k+1,0}(z), \ldots, x_{n-k+1,m-1}(z)\right)\right]
\end{aligned}$$

for $n = 1, 2, \dots$. Hence,

$$\begin{aligned} x^{(3+n)}(s) &= 2\sum_{k=0}^{n} \left[C_n^k P_{m,k}\left(x_{10}(s), \dots, x_{1,m-1}(s); \dots; x_{k0}(s), \dots, x_{k,m-1}(s)\right)\right. \\ &\quad \left.\times\, P_{m,n-k+1}\left(x_{10}(s), \dots, x_{1,m-1}(s); \dots; x_{n-k+1,0}(s), \dots, x_{n-k+1,m-1}(s)\right)\right] \\ &= 2\sum_{k=0}^{n} \left[C_n^k P_{m,k}\left(x'(s), \dots, x'(s); \dots; x^{(k)}(s), \dots, x^{(k)}(s)\right)\right. \\ &\quad \left.\times\, P_{m,n-k+1}\left(x'(s), \dots, x'(s); \dots; x^{(n-k+1)}(s), \dots, x^{(n-k+1)}(s)\right)\right] \\ &= 2\sum_{k=0}^{n} \left[C_n^k P_{m,k}\left(\mu, \dots, \mu; \dots; x^{(k)}(s), \dots, x^{(k)}(s)\right)\right. \\ &\quad \left.\times\, P_{m,n-k+1}\left(\mu, \dots, \mu; \dots; x^{(n-k+1)}(s), \dots, x^{(n-k+1)}(s)\right)\right] \\ &=: \lambda_n, \end{aligned}$$

for $n = 1, 2, \dots$. It is then easy to write out the explicit form of our solution $x(z)$:

$$x(z) = s + \mu(z-s) + \frac{s^2}{2!}(z-s)^2 + \frac{2s\mu^m}{3!}(z-s)^3 + \sum_{n=1}^{\infty} \frac{\lambda_n}{(n+3)!}(z-s)^{n+3}.$$

THEOREM 2. *Equation (1) has an analytic solution of the form*

$$x(z) = s^{(1-\beta)} z^{\beta} \tag{16}$$

in the region $|z - s| < s$ *satisfying*

$$x(s) = s, \qquad x'(s) = \beta, \tag{17}$$

where $s = \sqrt[(3)]{\beta(\beta-1)}$ *and* β *is a root of algebraic equation*

$$2\beta^m - \beta + 2 = 0. \tag{18}$$

PROOF. Let us seek for a solution of (1) in the form

$$x(z) = \lambda z^{\beta}. \tag{19}$$

Substituting (19) into (1) and comparing the coefficients, we obtain

$$\begin{aligned} \lambda\beta(\beta-1) &= \lambda^{2\left(1+\beta+\dots+\beta^{m-1}\right)}, \\ \beta - 2 &= 2\beta^m, \end{aligned}$$

that is,

$$\lambda = [\beta(\beta-1)]^{(1/3)(1-\beta)} = s^{1-\beta}, \tag{20}$$

$$2\beta^m - \beta + 2 = 0. \tag{21}$$

Therefore, in view of (20) and (21), we see that $x(z) = s^{1-\beta} z^{\beta}$ is a solution of (1) and show easily that (17). In fact, we have

$$\begin{aligned} x(s) &= \lambda s^{\beta} = s^{1-\beta} \cdot s^{\beta} = s, \\ x'(s) &= \beta\lambda s^{\beta-1} = \beta s^{1-\beta} s^{\beta-1} = \beta. \end{aligned}$$

Moreover, since

$$\begin{aligned}
x(z) &= s^{1-\beta} z^{\beta} = s^{1-\beta} \cdot s^{\beta} \left(1 + \frac{z-s}{s}\right)^{\beta} \\
&= s\left(1 + \frac{z-s}{s}\right)^{\beta} = s\left[1 + \beta\left(\frac{z-s}{s}\right)\right. \\
&\quad + \frac{\beta(\beta-1)}{2!}\left(\frac{z-s}{s}\right)^2 \\
&\quad \left. + \cdots + \frac{\beta(\beta-1)\ldots(\beta-n+1)}{n!}\left(\frac{z-s}{s}\right)^n + \cdots\right] \\
&= s + \beta(z-s) + \frac{\beta(\beta-1)}{2!s}(z-s)^2 \\
&\quad + \cdots + \frac{\beta(\beta-1)\ldots(\beta-n+1)}{n!s^{n-1}}(z-s)^n + \cdots,
\end{aligned}$$

we see that (16) is analytic in the region $|z-s| < s$. The proof is complete. $\blacksquare$

We remark that the equation $2\beta^2 - \beta + 2 = 0$ has two roots

$$\beta_{\pm} = \frac{1 \pm i\sqrt{15}}{4},$$

if $m = 2$ in (21). Thus, equation (1) has two analytic solutions

$$x_{\pm}(z) = [\beta_{\pm}(\beta_{\pm}-1)]^{(1/3)(1-\beta_{\pm})} z^{\beta_{\pm}} = \left(\frac{-9 \mp i\sqrt{15}}{8}\right)^{(3\mp i\sqrt{15})/12} z^{(1\pm i\sqrt{15})/4},$$

such that

$$x_{\pm}\left([\beta_{\pm}(\beta_{\pm}-1)]^{1/3}\right) = [\beta_{\pm}(\beta_{\pm}-1)]^{1/3}$$

and

$$\left|x'_{\pm}\left([\beta_{\pm}(\beta_{\pm}-1)]^{1/3}\right)\right| = |\beta_{\pm}| = 1,$$

$\beta_{\pm}$ is not a root of unity.

If $m = 3$ in (21), the equation $2\beta^3 - \beta + 2 = 0$ has three roots

$$\begin{aligned}
\beta_1 &= -\frac{1}{6}\left(\sqrt[3]{108 - 6\sqrt{318}} + \sqrt[3]{108 + 6\sqrt{318}}\right), \\
\beta_2 &= -\frac{1}{6}\left(\varepsilon_1\sqrt[3]{108 - 6\sqrt{318}} + \varepsilon_2\sqrt[3]{108 + 6\sqrt{318}}\right), \\
\beta_3 &= -\frac{1}{6}\left(\varepsilon_2\sqrt[3]{108 - 6\sqrt{318}} + \varepsilon_1\sqrt[3]{108 + 6\sqrt{318}}\right),
\end{aligned}$$

where

$$\varepsilon_1 = \frac{-1 + i\sqrt{3}}{2}, \qquad \varepsilon_2 = \frac{-1 - i\sqrt{3}}{2}.$$

Thus, equation (1) has three analytic solutions

$$x_j(z) = s_j^{1-\beta_j} z^{\beta_j}\left(s_j = [\beta_j(\beta_j - 1)]^{1/3},\ j = 1, 2, 3\right), \tag{22}$$

such that $x_j(s_j) = s_j (j = 1, 2, 3)$ and $|x'_1(s_1)| = |\beta_1| > 1$, $|x'_j(s_j)| = |\beta_j| < 1 (j = 2, 3)$. Obviously, the solution $x_1(z) = s_1^{1-\beta_1} z^{\beta_1}$ is not included in Theorem 1.

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